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# Weak order for the discretization of the stochastic heat equation.

Arnaud Debussche\*

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## Abstract

In this paper we study the approximation of the distribution of  $X_t$  Hilbert-valued stochastic process solution of a linear parabolic stochastic partial differential equation written in an abstract form as

$$dX_t + AX_t dt = Q^{1/2} dW_t, \quad X_0 = x \in H, \quad t \in [0, T],$$

driven by a Gaussian space time noise whose covariance operator  $Q$  is given. We assume that  $A^{-\alpha}$  is a finite trace operator for some  $\alpha > 0$  and that  $Q$  is bounded from  $H$  into  $D(A^\beta)$  for some  $\beta \geq 0$ . It is not required to be nuclear or to commute with  $A$ .

The discretization is achieved thanks to finite element methods in space (parameter  $h > 0$ ) and implicit Euler schemes in time (parameter  $\Delta t = T/N$ ). We define a discrete solution  $X_h^n$  and for suitable functions  $\varphi$  defined on  $H$ , we show that

$$|\mathbb{E} \varphi(X_h^N) - \mathbb{E} \varphi(X_T)| = O(h^{2\gamma} + \Delta t^\gamma)$$

where  $\gamma < 1 - \alpha + \beta$ . Let us note that as in the finite dimensional case the rate of convergence is twice the one for pathwise approximations.

**MSC classification:** 35A40, 60H15, 60H35, 65C30, 65M60

**Keywords:** Weak order, stochastic heat equation, finite element, Euler scheme.

## 1 Introduction

In this article, we study the convergence of the distributions of numerical approximations of the solutions of a large class of linear parabolic stochastic partial differential equations. The numerical analysis of stochastic partial differential equations has been recently the subject of many articles. (See among others [1], [8], [11], [12], [13], [14], [15], [16], [17], [18], [19], [24], [25], [28], [29], [35], [33], [34]). In all these papers, the aim is to give estimate on the strong order of convergence for a numerical scheme. In other words, on the order of pathwise convergence. It is well known that in the case of stochastic differential equations in finite dimension, the so-called weak order is much better than the strong order. The weak order is the order of convergence of the law of approximations to the true solution. For instance, the standard Euler scheme is of strong order 1/2 for the approximation of a

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stochastic differential equation while the weak order is 1. A basic tool to study the weak order is the Kolmogorov equation associated to the stochastic equation (see [22], [26], [27] [31]).

In infinite dimension, this problem has been studied in much less articles. In [3], the case of a stochastic delay equations is studied. To our knowledge, only [9], [20] consider stochastic partial differential equations. In [9] the nonlinear Schrödinger equations is considered. In the present article, we consider the case of the full discretization of a parabolic equation. We restrict our attention to a linear equation with additive noise which contains several difficulties. The general case of semilinear equations with state dependent noise present further difficulties and will be treated in a forthcoming article. This case is treated in [20] but there only finite dimensional functional of the solution are used and the finite dimensional method can be used.

Note that there are essential differences between the equations treated in [3] and [9]. Indeed, no spatial difference operator appear in a delay equation. In the case of the Schrödinger equation the linear evolution defines a group and it is possible to get rid of the differential operator by inverting the group. Furthermore, in [9], the data are assumed to be very smooth.

In this article, we get rid of the differential operator by a similar trick as in [9]. However, since the linear evolution operator is not invertible, this introduces extra difficulties. Moreover, we consider a full discretization using an implicit Euler scheme and finite elements for the spatial discretization. We give estimate of the weak order of convergence with minimal regularity assumptions on the data. In fact we show that as in the finite dimensional case the weak order is twice the strong order of convergence, both in time and space.

In dimension  $d = 1, 2, 3$ , let us consider the following stochastic partial differential equation:

$$(1.1) \quad \frac{\partial u(x, t)}{\partial t} - \Delta u(x, t) = \dot{\eta}(x, t),$$

where  $x \in \mathcal{O}$ , a bounded open set of  $\mathbb{R}^d$ , and  $t \in ]0, T]$ , with Dirichlet boundary conditions and initial data and  $\dot{\eta} = \frac{\partial \eta}{\partial t}$  with  $\eta$  denotes a real valued Gaussian process. It is convenient to use an abstract framework to describe the noise more precisely. Let  $W$  be a cylindrical Wiener process on  $L^2(\mathcal{O})$ , in other words  $\frac{\partial W}{\partial t}$  is the space time white noise. Equivalently, given an orthonormal basis of  $L^2(\mathcal{O})$ ,  $W$  has the following expansion

$$W(t) = \sum_{i \in \mathbb{N}} \beta_i(t) e_i$$

where  $(\beta_i)_{i \in \mathbb{N}}$  is a family of independent standard brownian motions (See section 2.4 below). We consider noises of the form  $\eta(t) = Q^{1/2} W(t)$  where  $Q$  is a non negative symmetric bounded linear operator on  $L^2(\mathcal{O})$ . For example<sup>1</sup>, given a function  $q$  defined on  $\mathcal{O}$ , we can take

$$\eta(x, t) = \int_{\mathcal{O}} q(x - y) W(y, t) dy,$$

Then the process  $\eta$  has the following correlation function:

$$\mathbb{E} \eta(x, t) \eta(y, s) = c(x - y)(t \wedge s) \quad \text{with } c(r) = \int_{\mathcal{O}} q(z + r) q(z) dz.$$

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<sup>1</sup>The equations describing this example are formal, it is not difficult to give a rigorous meaning. This is not important in our context. Note that  $q$  does not need to be a function and a distribution is allowed.

The operator  $Q$  is then given by  $Qf(x) = \int_{\mathcal{O}} c(x-y)f(y) dy$ . Note that if the  $q$  is the Dirac mass at 0,  $\eta = W$  and  $Q = I$ .

Let us also set  $A = -\Delta$ ,  $D(A) = H^2(\mathcal{O}) \cap H_0^1(\mathcal{O})$  and  $H = L^2(\mathcal{O})$ . Then  $A : D(A) \rightarrow H$  can be seen as an unbounded operator on  $H$  with domain  $D(A)$ . Our main assumption concerning  $Q$  is that  $A^{\sigma/2}Q$  defines a bounded operator on  $L^2(\mathcal{O})$  with  $\sigma > -1/2$  if  $d = 1$ ,  $\sigma > 0$  if  $d = 2$  and  $\sigma > 1/2$  if  $d = 3$ . In the example above this amounts to require that  $(-\Delta)^{\sigma/2}q \in L^2(\mathcal{O})$ . It is well known that these conditions are sufficient to ensure the existence of continuous solutions of (1.1).

If we write  $u(t) = u(\cdot, t)$  seen as a  $H$ -valued stochastic process then (1.1) can be rewritten under the abstract Ito form

$$(1.2) \quad du(t) + Au(t) dt = Q^{1/2} dW(t).$$

In this article, we consider such an abstract equation and study the approximation of the law of the solutions of (1.2) by means of finite elements of the distribution of  $u$  in  $H$ . Let  $\{V_h\}_{h \geq 0}$  be a family of finite dimensional subspaces of  $D(A^{1/2})$ . Let  $N \geq 1$  an integer and  $\Delta t = T/N$ . The numerical scheme is given by

$$(1.3) \quad (u_h^{n+1} - u_h^n, v_h) + \Delta t (Au_h^{n+\theta}, v_h) = \sqrt{\Delta t} (Q^{1/2} \chi^{n+1}, v_h),$$

for any  $v_h \in V_h$ , where  $\sqrt{\Delta t} \chi^{n+1} = W((n+1)\Delta t) - W(n\Delta t)$  is the noise increment and where  $(\cdot, \cdot)$  is the inner product of  $H$ . The unknown is approximated at time  $n\Delta t$ ,  $0 \leq n \leq N$  by  $u_h^n \in V_h$ . In (1.3), we have used the notation  $u_h^{n+\theta}$  for  $\theta u_h^{n+1} + (1-\theta)u_h^n$  for some  $\theta \in [0, 1]$ . We prove that an error estimate of the following form

$$|\mathbb{E}(\varphi(u(n\Delta t)) - \mathbb{E}(\varphi(u_h^n)))| \leq c(h^{2\gamma} + \Delta t^\gamma)$$

for any function  $\varphi$  which is  $C^2$  and bounded on  $H$ . With the above notation,  $\gamma$  is required to be strictly less than  $1 - d/2 + \sigma/2$ . This is exactly twice the strong order (see [33], [35]). If  $d = 1$  and  $\sigma = 0$ , the condition is  $\gamma < 1/2$  and we obtain a weak order which 1/2 in time and 1 in space.

## 2 Preliminaries

### 2.1 Functional spaces.

It is convenient to change the notations and rewrite the unknown of (1.2) as  $X$ . We thus consider the following stochastic partial differential equation written in the abstract form

$$(2.4) \quad dX_t + AX_t dt = Q^{1/2} dW_t, \quad X_0 = x \in H, \quad 0 < t \leq T,$$

where  $H$  is a Hilbert space whose inner product is denoted by  $(\cdot, \cdot)$  and its associated norm by  $|\cdot|$ , the process  $\{X_t\}_{t \in [0, T]}$  is an  $H$ -valued stochastic process,  $A$  a non negative self-adjoint unbounded operator on  $H$  whose domain  $D(A)$  is dense in  $H$  and compactly embedded in  $H$ ,  $Q$  a non negative symmetric operator on  $H$  and  $\{W_t\}_{t \in [0, T]}$  a cylindrical Wiener process on  $H$  adapted to a given normal filtration  $\{\mathcal{F}_t\}_{t \in [0, T]}$  in a given probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ .

It is well known that there exists a sequence of nondecreasing positive real numbers  $\{\lambda_n\}_{n \geq 1}$  together with  $\{e_n\}_{n \geq 1}$  a Hilbertian basis of  $H$  such that

$$Ae_n = \lambda_n e_n \quad \text{with} \quad \lim_{n \rightarrow +\infty} \lambda_n = +\infty.$$

We set for any  $s \geq 0$ ,

$$D(A^{s/2}) = \left\{ u = \sum_{n=1}^{+\infty} u_n e_n \in H \quad \text{such that} \quad \sum_{n=1}^{+\infty} \lambda_n^s u_n^2 < +\infty \right\},$$

and

$$A^s u = \sum_{n \geq 1} \lambda_n^s u_n e_n, \quad \forall u \in D(A^s).$$

It is clear that  $D(A^{s/2})$  endowed with the norm  $u \mapsto \|u\|_s := |A^{s/2}u|$  is a Hilbert space. We define also  $D(A^{-s/2})$  with  $s \geq 0$  as the completed space of  $H$  for the topology induced by the norm  $\|u\|_{-s} = \sum_{n \geq 1} \lambda_n^{-s} u_n^2$ . In this case  $D(A^{-s/2})$  can be identified with the topological dual of  $D(A^{s/2})$ , i.e. the space of the linear forms on  $D(A^{s/2})$  which are continuous with respect to the topology induced by the norm  $\|\cdot\|_s$ .

Moreover, these spaces can be obtained by interpolation between them. Indeed, for any reals  $s_1 \leq s \leq s_2$ , one has the continuous embeddings  $D(A^{s_1/2}) \subset D(A^{s/2}) \subset D(A^{s_2/2})$  and by Hölder inequality

$$(2.5) \quad \|u\|_s \leq \|u\|_{s_1}^{1-\lambda} \|u\|_{s_2}^\lambda, \quad s = (1-\lambda)s_1 + \lambda s_2,$$

for any  $u \in D(A^{s_2/2})$ .

We denote by  $\|\cdot\|_X$  the norm of a Banach space  $X$ . If  $X$  and  $Y$  denote two Banach spaces, we denote by  $\mathcal{L}(X, Y)$  the Banach space of bounded linear operators from  $X$  into  $Y$  endowed with the norm  $\|B\|_{\mathcal{L}(X, Y)} = \sup_{x \in X} \|Bx\|_Y / \|x\|_X$ . When  $X = Y$ , we use the shorter notation  $\mathcal{L}(X)$ .

If  $L \in \mathcal{L}(H)$  is a nuclear operator,  $\text{Tr}(L)$  denotes the trace of the operator  $L$ , i.e.

$$\text{Tr}(L) = \sum_{i \geq 1} (L e_i, e_i) \leq +\infty.$$

It is well known that the previous definition does not depend on the choice of the Hilbertian basis. Moreover, the following properties hold

$$(2.6) \quad \text{Tr}(LM) = \text{Tr}(ML), \quad \text{for any } L, M \in \mathcal{L}(H),$$

and

$$(2.7) \quad \text{Tr}(LM) \leq \text{Tr}(L) \|M\|_{\mathcal{L}(H)}, \quad \text{for any } L \in \mathcal{L}_+(H), M \in \mathcal{L}(H),$$

where  $\mathcal{L}_+(H)$  denotes the set non negative bounded linear operators on  $H$ .

Hilbert-Schmidt operators play also an important role. Given two Hilbert spaces  $K_1, K_2$ , an operator  $L \in \mathcal{L}(K_1, K_2)$  is Hilbert-Schmidt if  $L^*L$  is a nuclear operator on  $K_1$  or equivalently if  $LL^*$  is nuclear on  $K_2$ . We denote by  $\mathcal{L}_2(K_1, K_2)$  the space of such operators. It is a Hilbert space for the norm

$$\|L\|_{\mathcal{L}_2(K_1, K_2)} = (\text{Tr}(L^*L))^{1/2} = (\text{Tr}(LL^*))^{1/2}.$$

It is classical that, given four Hilbert spaces  $K_1, K_2, K_3, K_4$ , if  $L \in \mathcal{L}_2(K_2, K_3)$ ,  $M \in \mathcal{L}(K_1, K_2)$ ,  $N \in \mathcal{L}(K_3, K_4)$  then  $NLM \in \mathcal{L}_2(K_1, K_4)$  and

$$(2.8) \quad \|NLM\|_{\mathcal{L}_2(K_1, K_4)} \leq \|N\|_{\mathcal{L}(K_3, K_4)} \|L\|_{\mathcal{L}_2(K_2, K_3)} \|M\|_{\mathcal{L}(K_1, K_2)}.$$

See [6], appendix C, or [10] for more details on nuclear and Hilbert-Schmidt operators.

If  $X$  is a Banach space, we denote by  $\mathcal{C}_b(H; X)$  the Banach space of  $X$ -valued, continuous and bounded functions on  $H$ . We also denote by  $C_b^k(H)$  the space of  $k$ -times continuously differentiable real valued functions on  $H$ . The first order differential of a function  $\varphi \in C_b^1(H)$  is identified with its gradient and is then considered as an element of  $C_b(H; H)$ . It is denoted by  $D\varphi$ . Similarly, the second order differential of a function  $\varphi \in C_b^2(H)$  is seen as a function from  $H$  into the Banach space  $\mathcal{L}(H)$  and is denoted by  $D^2\varphi$ .

## 2.2 The deterministic stationary problem

We need some classical results on the deterministic stationary version of (2.4). In this case, special attention has to be paid to the space  $V = D(A^{1/2}) \subset H$ . It is a Hilbert space whose embedding into  $H$  is dense and continuous. Its inner product is denoted by  $((\cdot, \cdot))$ . We have

$$\begin{aligned} ((u, v)) &= (A^{1/2}u, A^{1/2}v), \quad \text{for any } u \in V, v \in V \\ &= (Au, v), \quad \text{for any } u \in D(A), v \in H. \end{aligned}$$

Then by a density argument and the uniqueness of the Riesz representation (in  $V$ ) we conclude that  $A$  is invertible from  $V$  into  $V' = D(A^{-1/2})$  or from  $D(A)$  into  $H$ . We will set  $T = A^{-1}$  its inverse. It is bounded and positive on  $H$  and on  $V$ .

For any  $f \in H$ ,  $u = Tf$  is by definition the unique solution of the following problem

$$(2.9) \quad u \in V, \quad ((u, v)) = (f, v), \quad \text{for any } v \in V.$$

Let  $\{V_h\}_{h>0}$  be a family of finite dimensional subspaces of  $V$  parametrized by a small parameter  $h > 0$ . For any  $h > 0$ , we denote by  $P_h$  (resp.  $\Pi_h$ ) the orthogonal projector from  $H$  (resp.  $V$ ) onto  $V_h$  with respect to the inner product  $(\cdot, \cdot)$  (resp.  $((\cdot, \cdot))$ ).

For any  $h > 0$ , we denote by  $A_h$  the linear bounded operator from  $V_h$  into  $V_h$  defined by

$$(2.10) \quad ((u_h, v_h)) = (A_h u_h, v_h) = (A u_h, v_h) \quad \text{for any } u_h \in V_h, v_h \in V_h.$$

It is clear that  $A_h : V_h \rightarrow V_h$  is also invertible. Its inverse is denoted by  $T_h$ . For any  $f \in H$ ,  $u_h = T_h f$  is by definition the solution of the following problem :

$$(2.11) \quad u_h \in V_h, \quad ((u_h, v_h)) = (f, v_h) = (P_h f, v_h), \quad \text{for any } v_h \in V_h.$$

It is also clear that  $A_h$  and  $T_h$  are positive definite symmetric bounded linear operators on  $V_h$ . We denote by  $\{\lambda_{i,h}\}_{1 \leq i \leq I(h)}$  the sequence of its nonincreasing positive eigenvalues and  $\{e_{i,h}\}_{1 \leq i \leq I(h)}$  the associated orthonormal basis of  $V_h$  of its eigenvectors. Again, by Hölder inequality,  $A_h$  satisfies the following interpolation inequality

$$(2.12) \quad |A_h^s u_h| \leq |A_h^{s_1} u_h|^\lambda |A_h^{s_2} u_h|^{1-\lambda}, \quad u_h \in V_h, \quad s = \lambda s_1 + (1 - \lambda) s_2.$$

The consequences of (2.10) are summarized in the following Lemma.

**Lemma 2.1** *Let  $A_h \in \mathcal{L}(V_h)$  defined in (2.10). Let  $T$  and  $T_h$  defined in (2.9) and (2.11). Then the following hold for any  $w_h \in V_h$  and  $v \in V$ :*

$$(2.13) \quad T_h P_h = \Pi_h T,$$

$$(2.14) \quad |A_h^{1/2} w_h| = |A_h^{1/2} w_h|$$

$$(2.15) \quad |T_h^{1/2} w_h| = |T^{1/2} w_h|,$$

$$(2.16) \quad |T_h^{1/2} P_h v| \leq |T^{1/2} v|.$$

### Proof

Now let  $f \in H$ . We consider the two solutions  $u$  and  $u_h$  of (2.9) and (2.11). Since  $V_h \subset V$ , we can write (2.9) with  $v_h \in V_h$ . Then, subtracting we get  $((u - u_h, v_h)) = 0$ . Hence,  $u_h = \Pi_h u$  the  $V$ -orthogonal projection of  $u$  onto  $V_h$ , i.e.  $T_h P_h f = \Pi_h T f$ .

Equation (2.14) follows immediately from the definition (2.10) of  $A_h$ . We now prove (2.16). Equations (2.13) and (2.14) imply  $|A_h^{-1/2} P_h v| = \|T_h P_h v\| = \|\Pi_h T v\| \leq \|T v\| = |A^{-1/2} v|$ , since  $\Pi_h : V \rightarrow V_h$  is an orthogonal projection for the inner product  $\|\cdot\|$ .

As regards (2.15), on one hand (2.16) with  $v = w_h \in V_h \subset V$  gives the first inequality  $|T_h^{1/2} w_h| \leq |T^{1/2} w_h|$ . On the other hand, by (2.10),

$$|(A_h u_h, v_h)| = |(A u_h, v_h)| \leq |A^{1/2} u_h| |A^{1/2} v_h| = |A^{1/2} u_h| |A^{1/2} v_h|.$$

So  $A_h u_h$  can be considered as a continuous linear form on  $D(A^{1/2})$ , i.e. belongs to  $D(A^{-1/2})$ , and

$$|A^{-1/2} A_h u_h| \leq |A^{1/2} u_h| = |A_h^{1/2} u_h|.$$

Taking  $u_h = A_h^{-1} w_h$  gives  $|A^{-1/2} w_h| \leq |A_h^{-1/2} w_h|$ . Eq. (2.15) follows. ■

Our main assumptions concerning the spaces  $V_h$  is that the corresponding linear elliptic problem (2.11) admits an  $O(h^r)$  error estimates in  $H$  and  $O(h^{r-1})$  in  $V$  for some  $r \geq 2$ . It is classical to verify that these estimates hold if we suppose that  $\Pi_h$  satisfies for some constant  $\kappa_0 > 0$ ,

$$(2.17) \quad |\Pi_h v - v| \leq \kappa_0 h^s |A^{s/2} v|, \quad 1 \leq s \leq r,$$

$$(2.18) \quad |A^{1/2}(\Pi_h w - w)| \leq \kappa_0 h^{s'-1} |A^{s'/2} w|, \quad 1 \leq s' \leq r-1,$$

where  $v \in D(A^{s/2})$  and  $w \in D(A^{s'/2})$ .

Finite elements satisfying these conditions are for example  $P_k$  triangular elements on a polygonal domain or  $Q_k$  rectangular finite element on a rectangular domain provided  $k \geq 1$ . Approximation by splines can also be considered. (See [4], [30]).

### 2.3 The deterministic evolution problem

We recall now some results about the spatial discretization of the solution of the deterministic linear parabolic evolution equation:

$$(2.19) \quad \frac{\partial u(t)}{\partial t} + A u(t) = 0, \quad u(0) = y,$$

by the finite dimensional one

$$\frac{\partial u_h(t)}{\partial t} + A_h u_h(t) = 0, \quad u_h(0) = P_h y \in V_h.$$

It is well known that, under our assumptions, (2.19) defines a contraction semi-group on  $H$  denoted by  $S(t) = e^{-tA}$  for any  $t \geq 0$ . Its solution can be read as  $u(t) = S(t)y$  where  $t \geq 0$ . The main properties of  $S(t)$  (contraction, regularization) are summed up below:

$$(2.20) \quad |e^{-tA} x| \leq |x|, \quad \text{for any } x \in H,$$

and

$$(2.21) \quad |A^s e^{-tA} x| \leq C(s) t^{-s} |x|,$$

for any  $t > 0$ ,  $s \geq 0$  and  $x \in H$ . Such a property is based on the definition of  $A^s$  and the following well known inequality

$$(2.22) \quad \sup_{x \geq 0} x^\varepsilon e^{-tx} \leq C(\varepsilon) t^{-\varepsilon}, \quad \text{for any } t > 0.$$

In the same manner, we denote by  $S_h(t)$  or  $e^{-tA_h}$  the semi-group on  $V_h$  such that  $u_h(t) = S_h(t)P_h y$ , for any  $t \geq 0$ .

We have various types of convergence of  $u_h$  towards  $u$  depending on the regularity of the initial data  $y$ . The optimal rates of convergences remain the same as in the corresponding stationary problems (see (2.17)–(2.18)). The estimates are not uniform in time near  $t = 0$  since the regularization of  $S(t)$  is used to prove them. The following Lemma gives two classical properties needed in this article.

**Lemma 2.2** *Let  $r \geq 2$  be such that (2.17) and (2.18) hold and  $q, q', s, s'$  such that  $0 \leq s \leq q \leq r$ ,  $s' \geq 0$ ,  $1 \leq q' + s' \leq r - 1$  and  $q' < 2$ . Then there exists constants  $\kappa_i > 0$ ,  $i = 1, 2$  independent on  $h$  such that for any time  $t > 0$ , one has:*

$$(2.23) \quad \|S_h(t)P_h - S(t)\|_{\mathcal{L}(D(A^{s/2}), H)} \leq \kappa_1 h^q t^{-(q-s)/2},$$

$$(2.24) \quad \|S_h(t)P_h - S(t)\|_{\mathcal{L}(D(A^{s'/2}), D(A^{1/2}))} \leq \kappa_2 h^{q'+s'} t^{-(q'+1)/2}.$$

The proof of (2.23) can be found in [2] (see also [32], Theorem 3.5, p. 45). The proof of (2.24) can be found in [21], Theorem 4.1, p. 342 (with  $f = 0$ ). In fact, we use only  $s = s' = 0$  and  $q' = 1$  below.

## 2.4 Infinite dimensional stochastic integrals

In this section, we recall basic results on the stochastic integral with respect to the cylindrical Wiener process  $W_t$ . More details can be found for instance in [6].

It is well known that  $W_t$  has the following expansion

$$W_t = \sum_{i=1}^{+\infty} \beta_i(t) e_i,$$

where  $\{\beta_i\}_{i \geq 1}$  denotes a family of real valued mutually independent Brownian motions on  $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t \geq 0})$ . The sum does not converge in  $H$  and this reflects the bad regularity property of the cylindrical Wiener process. However, it converges a.s. and in  $L^p(\Omega; U)$ ,  $p \geq 1$ , for any space  $U$  such that  $H \subset U$  with a Hilbert-Schmidt embedding. If  $H = L^2(\mathcal{O})$ ,  $\mathcal{O} \subset \mathbb{R}^d$  open and bounded, we can take  $U = H^{-s}(\mathcal{O})$ ,  $s > d/2$ .

Such a Wiener process can be characterized by

$$\mathbb{E}(W_t, u)(W_s, v) = \min(t, s)(u, v)$$

for any  $t, s \geq 0$  and  $u, v \in H$ .



Given any predictable operator valued function  $t \mapsto \Phi(t)$ ,  $t \in [0, T]$ , it is possible to define  $\int_0^T \Phi(s) dW(s)$  in a Hilbert space  $K$  if  $\Phi$  takes values in  $\mathcal{L}_2(H, K)$  and  $\int_0^T \|\Phi(s)\|_{\mathcal{L}_2(H, K)}^2 ds < \infty$  a.s. In this case  $\int_0^T \Phi(s) dW(s)$  is a well defined random variable with values in  $K$  and

$$\int_0^T \Phi(s) dW(s) = \sum_{i=1}^{\infty} \int_0^T \Phi(s) e_i d\beta_i(s).$$

Moreover, if  $\mathbb{E} \left( \int_0^T \|\Phi(s)\|_{\mathcal{L}_2(H, K)}^2 ds \right) < \infty$ , then

$$\mathbb{E} \left( \int_0^T \Phi(s) dW(s) \right) = 0,$$

and

$$\mathbb{E} \left( \left( \int_0^T \Phi(s) dW(s) \right)^2 \right) = \mathbb{E} \left( \int_0^T \|\Phi(s)\|_{\mathcal{L}_2(H, K)}^2 ds \right).$$

We will consider below expressions of the form  $\int_0^t \psi(s) Q^{1/2} dW(s)$ . These are then square integrable random variables in  $H$  with zero average if

$$\mathbb{E} \left( \int_0^T \|\psi(s) Q^{1/2}\|_{\mathcal{L}_2(H, K)}^2 ds \right) = \mathbb{E} \int_0^T \text{Tr}(\psi^*(s) Q \psi(s)) ds < \infty.$$

The solution of equation (2.4) can be written explicitly in terms of stochastic integrals. In order that these are well defined, we assume throughout this paper that there exists real numbers  $\alpha > 0$  and  $\min(\alpha - 1, 0) \leq \beta \leq \alpha$  such that

$$(2.25) \quad \sum_{n=1}^{\infty} \lambda_n^{-\alpha} = \|A^{-\alpha/2}\|_{\mathcal{L}_2(H)}^2 = \text{Tr}(A^{-\alpha}) < +\infty.$$

and

$$(2.26) \quad A^\beta Q \in \mathcal{L}(H).$$

Condition (2.26) implies that  $Q$  is a bounded operator from  $H$  into  $D(A^\beta)$ . By interpolation, we deduce immediately that for any  $\lambda \in [0, 1]$ ,  $A^{\lambda\beta} Q^\lambda \in \mathcal{L}(H)$  and

$$(2.27) \quad \|A^{\lambda\beta} Q^\lambda\|_{\mathcal{L}(H)} \leq \|A^\beta Q\|_{\mathcal{L}(H)}^\lambda.$$

**Example 2.3** *If one considers the equations described in the introduction where  $A$  is the Laplace operator with Dirichlet boundary conditions, it is well known that (2.25) holds for  $\alpha > d/2$ .*

We have the following result.

**Proposition 2.4** *Assume that (2.25), (2.26) hold and*

$$(2.28) \quad 1 - \alpha + \beta > 0.$$

*Then there exists a unique Gaussian stochastic process which is the weak solution (in the PDE sense) of (2.4) continuous in time with values in  $L^2(\Omega, H)$ . It is given by the formula which holds a.s. in  $H$ :*

$$X_t = e^{-tA} x + \int_0^t e^{-(t-s)A} Q^{1/2} dW_s = e^{-tA} x + \sum_{i=1}^{+\infty} \left( \int_0^t e^{-(t-s)\lambda_i} d\beta_i(s) \right) Q^{1/2} e_i.$$

## Proof

By Theorem 5.4 p. 121 in [6], it is sufficient to see that the stochastic integral make sense in  $H$ , i.e.

$$\int_0^t \|e^{-(t-s)A} Q^{1/2}\|_{\mathcal{L}_2(H)}^2 ds = \int_0^t \text{Tr} \left( e^{-(t-s)A} Q e^{-(t-s)A} \right) ds < \infty,$$

for any  $t \in [0, T]$ . We use (2.8) to estimate the Hilbert-Schmidt norm:

$$\begin{aligned} \|e^{-(t-s)A} Q^{1/2}\|_{\mathcal{L}_2(H)} &\leq \|A^{\beta/2} Q^{1/2}\|_{\mathcal{L}(H)} \|A^{-\alpha/2}\|_{\mathcal{L}_2(H)} \|e^{-(t-s)A} A^{(-\beta+\alpha)/2}\|_{\mathcal{L}(H)} \\ &\leq c(t-s)^{-1/2(-\beta+\alpha)} \end{aligned}$$

by (2.21), (2.25) and (2.27). The conclusion follows since  $-\beta + \alpha < 1$ . ■

## 3 Weak convergence of an implicit scheme.

### 3.1 Setting of the problem and main result.

In this section, we state the weak approximation result on the full discretization of (2.4).

We first describe the numerical scheme. Let  $N \geq 1$  be an integer and  $\{V_h\}_{h>0}$  the family of finite element spaces introduced in Section 2. Let  $\Delta t = T/N$  and  $t_n = n \Delta t$ ,  $0 \leq n \leq N$ . For any  $h > 0$  and any integer  $n \leq N$ , we seek for  $X_h^n$ , an approximation of  $X_{t_n}$ , such that for any  $v_h$  in  $V_h$ :

$$(3.29) \quad (X_h^{n+1} - X_h^n, v_h) + \Delta t (A X_h^{n+\theta}, v_h) = (Q^{1/2} W_{t_{n+1}} - Q^{1/2} W_{t_n}, v_h),$$

with the initial condition

$$(3.30) \quad (X_h^0, v_h) = (x, v_h), \quad \forall v_h \in V_h,$$

where

$$X_h^{n+\theta} = \theta X_h^{n+1} + (1 - \theta) X_h^n,$$

with

$$(3.31) \quad 1/2 < \theta \leq 1.$$

Recall that for  $\theta \leq 1/2$ , the scheme is in general unstable and a CFL condition is necessary.

Then (3.29)–(3.30) can be rewritten as

$$(3.32) \quad X_h^{n+1} - X_h^n + \Delta t A_h X_h^{n+\theta} = \sqrt{\Delta t} P_h Q^{1/2} \chi^{n+1},$$

$$(3.33) \quad X_h^0 = P_h x,$$

where

$$\chi^{n+1} = \frac{1}{\sqrt{\Delta t}} (W_{(n+1)\Delta t} - W_{n\Delta t}),$$

and where we recall that  $P_h : H \rightarrow V_h$  is the  $H$ -orthogonal projector. Hence  $\{\chi^n\}_{n \geq 0}$  is a sequence of independent and identically distributed gaussian random variables. The main result of this paper is stated below.

**Theorem 3.1** *Let  $\varphi \in C_b^2(H)$ , i.e. a twice differentiable real valued functional defined on  $H$  whose first and second derivatives are bounded. Let  $\alpha > 0$  and  $\beta \geq 0$  be such that (2.25), (2.26) and (2.28) hold. Let  $T \geq 1$  and  $\{X_t\}_{t \in [0, T]}$  be the  $H$ -valued stochastic process solution of (2.4) given by Proposition 2.4. For any  $N \geq 1$ , let  $\{X_h^n\}_{0 \leq n \leq N}$  be the solution of the scheme (3.32)–(3.33). Then there exists a constant  $C = C(T, \varphi) > 0$  which does not depend on  $h$  and  $N$  such that for any  $\gamma < 1 - \alpha + \beta \leq 1$ , the following inequality holds*

$$(3.34) \quad |\mathbb{E} \varphi(X_h^N) - \mathbb{E} \varphi(X_T)| \leq C (h^{2\gamma} + \Delta t^\gamma),$$

where  $\Delta t = T/N \leq 1$ .

### 3.2 Proof of Theorem 3.1.

The scheme (3.32)–(3.33) can be rewritten as

$$(3.35) \quad X_h^n = S_{h, \Delta t}^n P_h x + \sqrt{\Delta t} \sum_{k=0}^{n-1} S_{h, \Delta t}^{n-k-1} (I + \theta \Delta t A_h)^{-1} P_h \chi^{k+1}, \quad 0 \leq n \leq N,$$

where we have set for any  $h > 0$  and  $N \geq 1$ :

$$S_{h, \Delta t} = (I + \theta \Delta t A_h)^{-1} (I - (1 - \theta) \Delta t A_h).$$

**Step 1:** We introduce discrete and semi-discrete auxiliary schemes which will be useful for the proof of Theorem 3.1.

First, for any  $h > 0$ , let  $\{X_h(t)\}_{t \in [0, T]}$  be the  $V_h$ -valued stochastic process solution of the following finite dimensional stochastic partial differential equation

$$dX_{h,t} + A_h X_{h,t} dt = P_h Q^{1/2} dW_t, \quad X_{h,0} = P_h x.$$

It is straightforward to see that  $X_{h,t}$  can be written as

$$(3.36) \quad X_{h,t} = S_h(t) P_h x + \int_0^t S_h(t-s) P_h Q^{1/2} dW_s.$$

The last stochastic integral is well defined since  $t \mapsto \text{Tr}((S_h(t) P_h Q^{1/2})^* (S_h(t) P_h Q^{1/2}))$  is integrable on  $[0, T]$ .

We introduce also the following  $V_h$ -valued stochastic process

$$Y_{h,t} = S_h(T-t) X_{h,t}, \quad t \in [0, T],$$

which is solution of the following drift-free finite dimensional stochastic differential equation

$$(3.37) \quad dY_{h,t} = S_h(T-t) P_h Q^{1/2} dW_t, \quad Y_{h,0} = S_h(T) P_h x.$$

Its discrete counterpart is given by

$$(3.38) \quad \begin{aligned} Y_h^n &= S_{h, \Delta t}^{N-n} X_h^n, \quad 0 \leq n \leq N, \\ &= S_{h, \Delta t}^N P_h x + \sqrt{\Delta t} \sum_{k=0}^{n-1} S_{h, \Delta t}^{N-k-1} (I + \theta \Delta t A_h)^{-1} P_h Q^{1/2} \chi^{k+1}. \end{aligned}$$

Eventually, we consider a time continuous interpolation of  $Y_h^n$  which is the  $V_h$ -valued  $\{\mathcal{F}_t\}_t$ -adapted stochastic process  $\tilde{Y}_{h,t}$  defined by

$$(3.39) \quad \tilde{Y}_{h,t} = S_{h,\Delta t}^N P_h x + \int_0^t \sum_{n=0}^{N-1} S_{h,\Delta t}^{N-n-1} (I + \theta \Delta t A_h)^{-1} \mathbf{1}_n(s) P_h Q^{1/2} dW_s,$$

where  $\mathbf{1}_n$  denotes the function  $\mathbf{1}_{[t_n, t_{n+1}[}$ .

It is easy to see that for any  $t \in [0, T]$  and  $n$  be such that  $t \in [t_n, t_{n+1}[$ , we have

$$\tilde{Y}_{h,t} = Y_h^n + S_{h,\Delta t}^{N-n-1} (I + \theta \Delta t A_h)^{-1} P_h Q^{1/2} (W_t - W_{t_n}).$$

**Step 2:** Splitting of the error.

Let now  $\varphi \in C_b^2(H)$ . The error  $\mathbb{E} \varphi(X_h^N) - \mathbb{E} \varphi(X_T)$  can be splitted into two terms:

$$(3.40) \quad \begin{aligned} \mathbb{E} \varphi(X_h^N) - \mathbb{E} \varphi(X_T) &= \mathbb{E} \varphi(X_h^N) - \mathbb{E} \varphi(X_{h,T}) + \mathbb{E} \varphi(X_{h,T}) - \mathbb{E} \varphi(X_T) \\ &= A + B. \end{aligned}$$

The term  $A$  contains the error due to the time discretization and will be estimated uniformly with respect to  $h$ . The term  $B$  contains the spatial error.

**Step 3:** Estimate of the time discretization error.

Let us now estimate the time error uniformly with respect to  $h$ . In order to do this, we consider the solution  $v_h : V_h \rightarrow \mathbb{R}$  of the following deterministic finite dimensional Cauchy problem:

$$(3.41) \quad \begin{cases} \frac{\partial v_h}{\partial t} = \frac{1}{2} \text{Tr} \left( (S_h(T-t) P_h Q^{1/2})^* D^2 v_h (S_h(T-t) P_h Q^{1/2}) \right), \\ v_h(0) = \varphi. \end{cases}$$

We have the following classical representation of the solution of (3.41) at any time  $t \in [0, T]$  and for any  $y \in V_h$ :

$$(3.42) \quad v_h(T-t, y) = \mathbb{E} \varphi \left( y + \int_t^T S_h(T-s) P_h Q^{1/2} dW_s \right).$$

It follows easily

$$(3.43) \quad \|v_h(t)\|_{C_b^2(H)} \leq \|\varphi\|_{C_b^2(H)}, \quad t \in [0, T].$$

Now, the estimate of the time error relies mainly on the comparison of Itô formula applied successively to  $t \mapsto v_h(T-t, Y_{h,t})$  and  $t \mapsto v_h(T-t, \tilde{Y}_{h,t})$ . First, by construction,  $t \mapsto v_h(T-t, Y_{h,t})$  is a martingale. Indeed, Itô formula gives

$$dv_h(T-t, Y_{h,t}) = \left( Dv_h(T-t, Y_{h,t}), S_h(T-t) P_h Q^{1/2} dW_t \right).$$

Therefore

$$v_h(T-t, Y_{h,t}) = v_h(T, S_h(T) P_h x) + \int_0^t \left( Dv_h(T-s, Y_{h,s}), S_h(T-s) P_h Q^{1/2} dW_s \right).$$

Taking  $t = T$  and the expectation implies

$$(3.44) \quad \mathbb{E} \varphi(X_{h,T}) = v_h(T, S_h(T) P_h x).$$

On the contrary,  $t \mapsto v_h(T-t, \tilde{Y}_{h,t})$  is not a martingale. Nevertheless, applying Itô formula gives, thanks to (3.39),

$$\begin{aligned} \mathbb{E} v_h(0, \tilde{Y}_{h,T}) &= \mathbb{E} v_h(T, \tilde{Y}_{h,0}) - \mathbb{E} \int_0^T \frac{\partial v_h}{\partial t}(T-t, \tilde{Y}_{h,t}) dt \\ (3.45) \quad &+ \frac{1}{2} \mathbb{E} \int_0^T \sum_{n=0}^{N-1} \text{Tr} \left[ \left( S_{h,\Delta t}^{N-n-1} T_{h,\Delta t} P_h Q^{1/2} \right)^* D^2 v_h \left( S_{h,\Delta t}^{N-n-1} T_{h,\Delta t} P_h Q^{1/2} \right) \right] \mathbf{1}_n(t) dt, \end{aligned}$$

where here and in equations (3.46), (3.47) below,  $D^2 v_h$  is evaluated at  $(T-t, \tilde{Y}_{h,t})$ . Also we have set

$$T_{h,\Delta t} = (I + \theta \Delta t A_h)^{-1}.$$

Now, plugging (3.41) into (3.45) gives:

$$\begin{aligned} \mathbb{E} \varphi(X_h^N) &= v_h(T, S_{h,\Delta t}^N P_h x) \\ (3.46) \quad &+ \frac{1}{2} \mathbb{E} \int_0^T \sum_{n=0}^{N-1} \text{Tr} \left[ \left( S_{h,\Delta t}^{N-n-1} T_{h,\Delta t} P_h Q^{1/2} \right)^* D^2 v_h \left( S_{h,\Delta t}^{N-n-1} T_{h,\Delta t} P_h Q^{1/2} \right) \right. \\ &\quad \left. - \left( S_h(T-t) P_h Q^{1/2} \right)^* D^2 v_h \left( S_h(T-t) P_h Q^{1/2} \right) \right] \mathbf{1}_n(t) dt. \end{aligned}$$

At last, the comparison between (3.44) and (3.46) leads to the following decomposition of the time error  $A$

$$\begin{aligned} \mathbb{E} \varphi(X_h^N) - \mathbb{E} \varphi(X_{h,T}) &= v_h(T, S_{h,\Delta t}^N P_h x) - v_h(T, S_h(T) P_h x) \\ (3.47) \quad &+ \frac{1}{2} \mathbb{E} \int_0^T \sum_{n=0}^{N-1} \text{Tr} \left[ \left( S_{h,\Delta t}^{N-n-1} T_{h,\Delta t} P_h Q^{1/2} \right)^* D^2 v_h \left( S_{h,\Delta t}^{N-n-1} T_{h,\Delta t} P_h Q^{1/2} \right) \right. \\ &\quad \left. - \left( S_h(T-t) P_h Q^{1/2} \right)^* D^2 v_h \left( S_h(T-t) P_h Q^{1/2} \right) \right] \mathbf{1}_n(t) dt, \\ &= I + II, \end{aligned}$$

The term  $I$  is the pure deterministic part of the time error. Thanks to the representation (3.42), we have

$$(3.48) \quad I \leq \|\varphi\|_{\mathcal{C}_b^1(H)} \|S_h(T) P_h - S_{h,\Delta t}^N P_h\|_{\mathcal{L}(H)} |x|.$$

Thanks to (3.31) it is possible to bound  $I$  uniformly with respect to  $h$ . More precisely, we have

$$\begin{aligned} (3.49) \quad \|(S_h(N\Delta t) - S_{h,\Delta t}^N) P_h\|_{\mathcal{L}(H)} &= \sup_{i \geq 1} \left| e^{-N\lambda_{i,h}\Delta t} - F^N(\lambda_{i,h}\Delta t) \right| \\ &\leq \sup_{z \geq 0} |e^{-Nz} - F^N(z)| \\ &\leq \frac{\kappa_4}{N} \leq \kappa_4 \Delta t, \end{aligned}$$

for any  $T \geq 1$  (e.g. see Theorem 1.1, p. 921 in [23]). We have used the following notation:

$$F(z) = \frac{1 - (1 - \theta)z}{1 + \theta z}, \quad z > 0.$$

Let us now see how to estimate the term  $II$ . First, using the symmetry of  $D^2v_h$ , we rewrite the trace term as

$$\begin{aligned} & \text{Tr} \left( \left( S_{h,\Delta t}^{N-n-1} T_{h,\Delta t} P_h Q^{1/2} - S_h(T-t) P_h Q^{1/2} \right)^* D^2v_h \left( S_{h,\Delta t}^{N-n-1} T_{h,\Delta t} P_h Q^{1/2} - S_h(T-t) P_h Q^{1/2} \right) \right) \\ & + 2 \text{Tr} \left( \left( S_h(T-t) P_h Q^{1/2} \right)^* D^2v_h \left( S_{h,\Delta t}^{N-n-1} T_{h,\Delta t} P_h Q^{1/2} - S_h(T-t) P_h Q^{1/2} \right) \right) \\ & = a_n(t) + b_n(t). \end{aligned}$$

Let now  $\alpha > 0$  and  $\beta \geq 0$  such that (2.25) and (2.26) hold with  $0 < 1 - \alpha + \beta \leq 1$ . Let  $\gamma > 0$  and  $\gamma_1 > 0$  such that  $0 < \gamma < \gamma_1 < 1 - \alpha + \beta \leq 1$ .

We first estimate the term  $a_n(t)$ . We use (2.6), (2.7), (3.43) and (2.8) to obtain

$$\begin{aligned} a_n(t) & \leq \|D^2v_h(T-t)\|_{C_b^2(H)} \\ & \quad \times \text{Tr} \left( \left( S_{h,\Delta t}^{N-n-1} T_{h,\Delta t} P_h Q^{1/2} - S_h(T-t) P_h Q^{1/2} \right)^* \left( S_{h,\Delta t}^{N-n-1} T_{h,\Delta t} P_h Q^{1/2} - S_h(T-t) P_h Q^{1/2} \right) \right) \\ & \leq \|\varphi\|_{C_b^2(H)} \left\| \left( S_{h,\Delta t}^{N-n-1} T_{h,\Delta t} - S_h(T-t) \right) P_h Q^{1/2} \right\|_{\mathcal{L}_2(H)}^2 \\ & \leq \|\varphi\|_{C_b^2(H)} \left\| \left( S_{h,\Delta t}^{N-n-1} T_{h,\Delta t} - S_h(T-t) \right) A_h^{(1-\gamma_1)/2} P_h \right\|_{\mathcal{L}(V_h)}^2 \|A_h^{(\gamma_1-1)/2} P_h Q^{1/2}\|_{\mathcal{L}_2(H,V_h)}^2. \end{aligned} \quad (3.50)$$

Note that here  $V_h$  is endowed with the norm of  $H$ . Let us set

$$\begin{aligned} M_n(t) & = \left\| \left( S_{h,\Delta t}^{N-n-1} T_{h,\Delta t} - S_h(T-t) \right) A_h^{(1-\gamma_1)/2} P_h \right\|_{\mathcal{L}(V_h)} \\ & = \sup_{1 \leq i \leq I(h)} \left| \frac{F^{N-n-1}(\lambda_{i,h} \Delta t)}{1 + \theta \lambda_{i,h} \Delta t} - e^{-\lambda_{i,h}(T-t)} \right| \lambda_{i,h}^{(1-\gamma_1)/2}. \end{aligned} \quad (3.51)$$

Using similar techniques as for the proof of the strong order (see e.g. [28]), we have the following bound, for  $n < N-1$

$$M_n(t) \leq C \Delta t^{\gamma/2} ((N-n-1)\Delta t)^{-(1-\gamma_1+\gamma)/2}, \quad (3.52)$$

where here and below  $C$  denotes a constant which depends only on  $\gamma_1$ ,  $\gamma$ ,  $\|\varphi\|_{C_b^2(H)}$ ,  $\|A^\beta Q\|_{\mathcal{L}(H)}$  and  $\text{Tr}(A^{-\alpha})$ . In particular these constants do not depend on  $h$  or  $\Delta t$ . The proof is postponed to the appendix at the end of this article.

We then estimate the last factor in (3.50). Since  $V_h \subset H$  and we have endowed  $V_h$  with the norm of  $H$ , we may write

$$\|A_h^{(\gamma_1-1)/2} P_h Q^{1/2}\|_{\mathcal{L}_2(H,V_h)} \leq \|A_h^{(\gamma_1-1)/2} P_h Q^{1/2}\|_{\mathcal{L}_2(H)}.$$

Using (2.8), we deduce

$$\|A_h^{(\gamma_1-1)/2} P_h Q^{1/2}\|_{\mathcal{L}_2(H,V_h)} \leq \|A_h^{(\gamma_1-1)/2} P_h A^{-\beta/2}\|_{\mathcal{L}_2(H)} \|A^{\beta/2} Q^{1/2}\|_{\mathcal{L}(H)}.$$

Using (2.12) and (2.16), we have

$$\begin{aligned} \|A_h^{(\gamma_1-1)/2} P_h A^{-\beta/2}\|_{\mathcal{L}_2(H)}^2 & = \sum_{i \in \mathbb{N}} |A_h^{(\gamma_1-1)/2} P_h A^{-\beta/2} e_i|^2 \\ & \leq \sum_{i \in \mathbb{N}} |A_h^{-1/2} P_h A^{-\beta/2} e_i|^{2(1-\gamma_1)} |P_h A^{-\beta/2} e_i|^{2\gamma_1} \\ & \leq \sum_{i \in \mathbb{N}} |A_h^{-1/2-\beta/2} e_i|^{2(1-\gamma_1)} |A^{-\beta/2} e_i|^{2\gamma_1} \\ & = \sum_{i \in \mathbb{N}} \lambda_i^{-(1-\gamma_1+\beta)}. \end{aligned}$$

We deduce from  $1 - \gamma_1 + \beta > \alpha$  and (2.27) that

$$\|A_h^{(\gamma_1-1)/2} P_h A^{-\beta/2}\|_{\mathcal{L}_2(H)}^2 \leq \lambda_1^{1-\gamma_1+\beta-\alpha} \|A^\beta Q\|_{\mathcal{L}(H)}^{1/2} \text{Tr}(A^{-\alpha})$$

Plugging this and (3.52) into (3.50) yields for  $n < N - 1$ :

$$(3.53) \quad a_n(t) \leq C \Delta t^\gamma ((N - n - 1) \Delta t)^{-(1-\gamma_1+\gamma)}.$$

For  $n = N - 1$ , we derive similarly,

$$(3.54) \quad \begin{aligned} a_{N-1}(t) &\leq \|\varphi\|_{C_b^2(H)} \|(T_{h,\Delta t} - S_h(T-t)) A_h^{(1-\gamma_1)/2}\|_{\mathcal{L}(V_h)}^2 \|A_h^{(1-\gamma_1)/2} P_h Q^{1/2}\|_{\mathcal{L}_2(H,V_h)}^2 \\ &\leq C \left( \|T_{h,\Delta t} A_h^{(1-\gamma_1)/2}\|_{\mathcal{L}(V_h)}^2 + \|S_h(T-t) A_h^{(1-\gamma_1)/2}\|_{\mathcal{L}(V_h)}^2 \right) \\ &\leq C (\Delta t^{\gamma_1-1} + (T-t)^{\gamma_1-1}) \\ &\leq C (T-t)^{\gamma_1-1}. \end{aligned}$$

Concerning  $b_n$ , we write

$$\begin{aligned} b_n(t) &= 2 \operatorname{Tr} \left( \left( S_h(T-t) A_h^{(1-\gamma_1)/2} A_h^{(\gamma_1-1)/2} P_h Q^{1/2} \right)^* D^2 v_h \right. \\ &\quad \left. \left( S_{h,\Delta t}^{N-n-1} T_{h,\Delta t} - S_h(T-t) \right) A_h^{(1-\gamma_1)/2} A_h^{(\gamma_1-1)/2} P_h Q^{1/2} \right) \\ &\leq \|\varphi\|_{C_b^2(H)} \|S_h(T-t) A_h^{(1-\gamma_1)/2}\|_{\mathcal{L}(V_h)} \left\| \left( S_{h,\Delta t}^{N-n-1} T_{h,\Delta t} - S_h(T-t) \right) A_h^{(1-\gamma_1)/2} \right\|_{\mathcal{L}(V_h)} \\ &\quad \times \|A_h^{(\gamma_1-1)/2} P_h Q^{1/2}\|_{\mathcal{L}_2(H,V_h)}^2 \end{aligned}$$

Using similar argument as above, we prove

$$\|S_h(T-t) A_h^{(1-\gamma_1)/2}\|_{\mathcal{L}(V_h)} \leq C (T-t)^{(\gamma_1-1)/2}$$

and, for  $n < N - 1$ ,

$$\left\| \left( S_{h,\Delta t}^{N-n-1} T_{h,\Delta t} - S_h(T-t) \right) A_h^{(1-\gamma_1)/2} \right\|_{\mathcal{L}(V_h)} \leq C \Delta t^\gamma ((N - n - 1) \Delta t)^{-(1-\gamma_1+\gamma)}$$

so that  $n < N - 1$ :

$$(3.55) \quad b_n(t) \leq C \Delta t^\gamma ((N - n - 1) \Delta t)^{-(1-\gamma_1+\gamma)}.$$

For  $n = N - 1$ , we have

$$(3.56) \quad \begin{aligned} b_{N-1}(t) &\leq \|\varphi\|_{C_b^2(H)} \|S_h(T-t) A_h^{(1-\gamma_1)/2}\|_{\mathcal{L}(V_h)} \\ &\quad \times \|(T_{h,\Delta t} - S_h(T-t)) A_h^{(1-\gamma_1)/2} P_h\|_{\mathcal{L}(V_h)} \|A_h^{(1-\gamma_1)/2} P_h Q^{1/2}\|_{\mathcal{L}_2(H,V_h)}^2 \\ &\leq C (T-t)^{\gamma_1-1} \end{aligned}$$

We are now ready to bound  $II$  in (3.47). Indeed, (3.53), (3.54), (3.55), (3.56) imply

$$(3.57) \quad \begin{aligned} II &\leq C \int_{\Delta t}^T \sum_{n=1}^{N-2} \Delta t^\gamma ((N - n - 1) \Delta t)^{-(1-\gamma_1+\gamma)} \mathbf{1}_n(t) dt + \int_0^{\Delta t} C (T-t)^{\gamma_1-1} dt \\ &\leq C \Delta t^\gamma \end{aligned}$$

Then, plugging (3.48) and (3.57) into (3.47) we obtain that

$$(3.58) \quad \begin{aligned} |A| &\leq \kappa_4 \|D\varphi\|_{C_b(H;H)} |x| \Delta t + \frac{C_3}{2} T^{\gamma_1-\gamma} \Delta t^\gamma \\ &\leq C \Delta t^\gamma, \end{aligned}$$

for  $T \geq 1$ ,  $\Delta t \leq 1$ .

**Step 4:** Estimate of the space discretization error.

Let us now estimate the spatial error  $B$ . The method is essentially the same as above: we use the Kolmogorov equation associated to the transformed process  $Y_t$ .

We consider the following linear parabolic equation on  $H$ :

$$(3.59) \quad \frac{\partial v}{\partial t}(t, x) = \frac{1}{2} \text{Tr} \left( D^2 v(t, x) (S(T-t)Q^{1/2})(S(T-t)Q^{1/2})^* \right), \quad t > 0, \quad x \in H,$$

together with the initial condition

$$v(0, x) = \varphi(x), \quad x \in H,$$

where  $v$  is a real-valued function of  $t$  and  $x \in H$ . We have the following representation of  $v$  (see e.g. [7], chapter 3) at time  $t \in [0, T]$  and any  $y \in H$ :

$$(3.60) \quad v(T-t, y) = \mathbb{E} \varphi \left( y + \int_t^T S(T-s) Q^{1/2} dW_s \right).$$

We apply the Itô formula to  $t \mapsto v(T-t, Y_t)$  and  $t \mapsto v(T-t, Y_{h,t})$ . We subtract the resulting equations and obtain

$$(3.61) \quad \begin{cases} \mathbb{E} \varphi(X_T) - \mathbb{E} \varphi(X_{h,T}) = v(T, S(T)x) - v(T, S_h(T)P_h x) \\ + \frac{1}{2} \mathbb{E} \int_0^T \text{Tr} \left[ \left( S_h(T-t)P_h Q^{1/2} \right)^* D^2 v(T-t, Y_{h,t}) \left( S_h(T-t)P_h Q^{1/2} \right) \right] dt \\ - \frac{1}{2} \mathbb{E} \int_0^T \text{Tr} \left[ \left( S(T-t)Q^{1/2} \right)^* D^2 v(T-t, Y_{h,t}) \left( S(T-t)Q^{1/2} \right) \right] dt. \end{cases}$$

The first term on the right hand side of (3.61) is the deterministic spatial error which can be bounded thanks to (2.23) (with  $s = 0$  and  $q = 2\gamma < 2$ ) and (3.60). We obtain:

$$(3.62) \quad |v(T, S(T)x) - v(T, S_h(T)P_h x)| \leq \kappa_1 \|\varphi\|_{C_b(H)} h^{2\gamma} T^{-\gamma} |x|.$$

For the second term, we use the symmetry of  $D^2 v$  and write

$$\begin{aligned} & \text{Tr} \left[ \left( S_h(T-t)P_h Q^{1/2} \right)^* D^2 v \left( S_h(T-t)P_h Q^{1/2} \right) - \left( S(T-t)Q^{1/2} \right)^* D^2 v \left( S(T-t)Q^{1/2} \right) \right] \\ &= \text{Tr} \left[ \left( S_h(T-t)P_h Q^{1/2} - S(T-t)Q^{1/2} \right)^* D^2 v \left( S_h(T-t)P_h Q^{1/2} - S(T-t)Q^{1/2} \right) \right] \\ &+ 2 \text{Tr} \left[ \left( S(T-t)Q^{1/2} \right)^* D^2 v \left( S_h(T-t)P_h Q^{1/2} - S(T-t)Q^{1/2} \right) \right] \\ &= a + b, \end{aligned}$$

where here and below  $D^2 v$  is evaluated at  $(T-t, Y_{h,t})$ .

Let  $\gamma > 0$  be such that (2.28) holds and  $\gamma_1 > 0$  such that  $0 < \gamma < \gamma_1 < 1 - \alpha + \beta \leq 1$ . Thanks to (2.7) and (3.60), we get the following bounds:

$$\begin{aligned} b &= 2 \text{Tr} \left( S(T-t) D^2 v(S_h(T-t)P_h - S(T-t))Q \right) \\ &= 2 \text{Tr} \left( A^{\gamma_1-1-\beta} A^{1-\gamma_1} S(T-t) D^2 v(S_h(T-t)P_h - S(T-t))Q A^\beta \right) \\ &\leq 2 \| (S_h(T-t)P_h - S(T-t)) \|_{\mathcal{L}(H)} \|\varphi\|_{C_b^2(H)} \|A^{1-\gamma_1} S(T-t)\|_{\mathcal{L}(H)} \|Q A^\beta\|_{\mathcal{L}(H)} \text{Tr}(A^{1-\gamma_1-\beta}). \end{aligned}$$



Then, owing to (2.21), (2.23) (with  $s = 0$  and  $q = 2\gamma < 2$ ), we obtain

$$(3.63) \quad b \leq Ch^{2\gamma}(T-t)^{-(1+\gamma-\gamma_1)}$$

where again  $C$  denotes a constant which depends only on  $\gamma_1$ ,  $\gamma$ ,  $\|\varphi\|_{C_b^2(H)}$ ,  $\|A^\beta Q\|_{\mathcal{L}(H)}$  and  $\text{Tr}(A^{-\alpha})$  but not on  $h$  or  $\Delta t$ .

As regard  $a$ , we get first thanks to (2.7) and (3.60):

$$\begin{aligned} a &\leq \|\varphi\|_{C_b^2(H)} \text{Tr} \left( Q(S_h(T-t)P_h - S(T-t))^* (S_h(T-t)P_h - S(T-t)) \right) \\ &= \|\varphi\|_{C_b^2(H)} \|Q^{1/2}(S_h(T-t)P_h - S(T-t))\|_{\mathcal{L}_2(H)}^2 \\ &\leq \|\varphi\|_{C_b^2(H)} \|Q^{1/2}A^{\beta/2}\|_{\mathcal{L}(H)}^2 \|A^{(1-\gamma_1)/2}(S_h(T-t)P_h - S(T-t))\|_{\mathcal{L}(H)}^2 \|A^{-(1-\gamma_1+\beta)/2}\|_{\mathcal{L}_2(H)}^2 \\ &\leq C \|A^{(1-\gamma_1)/2}(S_h(T-t)P_h - S(T-t))\|_{\mathcal{L}(H)}^2, \end{aligned}$$

where we have used (2.27). If  $\gamma_1 + \gamma \geq 1$ , we interpolate (2.23) with  $q = (\gamma + \gamma_1 - 1)/\gamma$ ,  $s = 0$  and (2.24) with  $s' = 0$ ,  $q' = 1$  and get

$$(3.64) \quad \|A^{(1-\gamma_1)/2}(S_h(T-t)P_h - S(T-t))\|_{\mathcal{L}(H)} \leq Ch^\gamma(T-t)^{-(1-\gamma_1+\gamma)/2}.$$

If  $\gamma_1 + \gamma < 1$ , we interpolate (2.23) with  $q = 0$ ,  $s = 0$  and (2.24) with  $s' = 0$ ,  $q' = 1$  and get

$$(3.65) \quad \|A^{(1-\gamma_1)/2}(S_h(T-t)P_h - S(T-t))\|_{\mathcal{L}(H)} \leq Ch^{1-\gamma_1}(T-t)^{-(1-\gamma_1)}.$$

We use again an interpolation argument to get

$$\begin{aligned} (3.66) \quad &\|A^{(1-\gamma_1)/2}(S_h(T-t)P_h - S(T-t))\|_{\mathcal{L}(H)} \\ &\leq C \|(S_h(T-t)P_h - S(T-t))\|_{\mathcal{L}(H)}^{\gamma_1} \|A^{1/2}(S_h(T-t)P_h - S(T-t))\|_{\mathcal{L}(H)}^{1-\gamma_1} \\ &\leq C (\|A^{1/2}S_h(T-t)P_h\|_{\mathcal{L}(H)} + \|A^{1/2}S(T-t)\|_{\mathcal{L}(H)})^{1-\gamma_1} \\ &\leq C(T-t)^{(1-\gamma_1)/2}, \end{aligned}$$

thanks to (2.21) for  $A$  and  $A_h$  and (2.14). A further interpolation between (3.65) and (3.66) gives

$$(3.67) \quad \|A^{(1-\gamma_1)/2}(S_h(T-t)P_h - S(T-t))\|_{\mathcal{L}(H)} \leq Ch^{\lambda(1-\gamma_1)}(T-t)^{-(1-\gamma_1)(\lambda+(1-\lambda)/2)}.$$

Taking  $\lambda = \gamma/(1-\gamma_1)$  shows that (3.64) again holds for  $\gamma_1 + \gamma \leq 1$ .

We deduce

$$(3.68) \quad a \leq Ch^{2\gamma}(T-t)^{-1+\gamma_1-\gamma}.$$

Plugging (3.62), (3.63) and (3.68) into (3.61) leads to, after time integration which is relevant since  $1 - \gamma_1 + \gamma < 1$ :

$$(3.69) \quad |B| \leq Ch^{2\gamma},$$

for  $T \geq 1$ .

**Conclusion:** Gathering (3.58) and (3.69) in (3.40) ends the proof of Theorem 3.1.

## References

- [1] E.J. ALLEN, S.J. NOVOSEL, Z. ZHANG, *Finite element and difference approximation of some linear stochastic partial differential equations*, Stochastics Rep. **64** (1998), n° 1-2, 117–142.
- [2] J. H. BRAMBLE, A. H. SCHATZ, V. THOMÉE, L. B. WAHLBIN, *Some convergence estimates for semidiscrete Galerkin type approximations for parabolic equations*, SIAM J. Numer. Anal. **14** (1977), n° 2, 218–241.
- [3] E. BUCKWAR, T. SHARDLOW, *Weak approximation of stochastic differential delay equations*, IMA J. Numer. Anal. **25** (2005), n° 1, 57–86.
- [4] P. G. CIARLET, THE FINITE ELEMENT METHOD FOR ELLIPTIC PROBLEMS, North-Holland, Amsterdam–London–New-York, 1978.
- [5] M. CROUZEIX, V. THOMÉE, L. B. WAHLBIN, *Error estimates for spatially discrete approximations of semilinear parabolic equations with initial data of low regularity*, Math. Comput. **53** (1989), 25–41.
- [6] G. DA PRATO and J. ZABCZYK, STOCHASTIC EQUATIONS IN INFINITE DIMENSIONS, Encyclopedia of mathematics and its applications **44**, Cambridge University Press, 1992.
- [7] G. DA PRATO and J. ZABCZYK, SECOND ORDER PARTIAL DIFFERENTIAL EQUATIONS IN HILBERT SPACES, London Mathematical Society, Lecture Note Series **293**, Cambridge University Press, 2002.
- [8] A.M. DAVIE, J.G. GAINES, *Convergence of numerical schemes for the solution of parabolic stochastic partial differential equations*, Math. Comp. **70** (2001), n° 233, 121–134.
- [9] A. DE BOUARD, A. DEBUSSCHE *Weak and strong order of convergence of a semi discrete scheme for the stochastic Nonlinear Schrodinger equation*, Applied Mathematics and Optimization Journal **54** (2006), n° 3, 369–399.
- [10] I. C. GOKHBERG, M. G. KREĬN, INTRODUCTION TO THE THEORY OF LINEAR NONSELFADJOINT OPERATORS IN HILBERT SPACE , Amer. Math. Soc., Providence, R. I., 1970.
- [11] W. GRECKSCH, P.E. KLOEDEN *Time-discretised Galerkin approximations of parabolic stochastic PDEs*, Bull. Austral. Math. Soc. **54** (1996), n° 1, 79–85.
- [12] I. GYONGY *Lattice approximations for stochastic quasi-linear parabolic partial differential equations driven by space-time white noise. I*, Potential Anal. **9** (1998), n° 1, 1–25.
- [13] I. GYONGY *Lattice approximations for stochastic quasi-linear parabolic partial differential equations driven by space-time white noise. II*, Potential Anal. **11** (1999), n° 1, 1–37.
- [14] I. GYONGY, A. MILLET *On discretization schemes for stochastic evolution equations*, Potential Analysis **23** (2005), n° 2, 99–134.
- [15] I. GYONGY, A. MILLET *Rate of Convergence of Implicit Approximations for stochastic evolution equations*, Stochastic Differential Equations: Theory and Applications. A volume in Honor of Professor Boris L. Rosovskii, Interdisciplinary Mathematical Sciences, Vol 2, World Scientific (2007), 281–310.
- [16] I. GYONGY, A. MILLET *Rate of Convergence of Space Time Approximations for stochastic evolution equations*, Preprint (2007).
- [17] I. GYONGY, D. NUALART *Implicit scheme for stochastic parabolic partial differential equations driven by space-time white noise*, Potential Anal. **7** (1997), n° 4, 725–757.
- [18] E. HAUSENBLAS, *Approximation for Semilinear Stochastic Evolution Equations in Banach Spaces*, Journal in Computational and Applied Mathematics, **147** (2002), no 2, 485–516.
- [19] E. HAUSENBLAS, *Approximation for Semilinear Stochastic Evolution Equations*, Potential Analysis, **18** (2003), no 2, 141–186.

- [20] E. HAUSENBLAS, *Weak Approximation of Stochastic Partial Differential Equations*. in Capar, U. and Üstünel, A., editor, *Stochastic analysis and related topics VIII. Silivri workshop*, Progress in Probability. Basel: Birkhäuser, 2003.
- [21] C. JOHNSON, S. LARSSON, V. THOMÉE, L. B. WALHBIN, *Error estimates for spatially discrete approximations of semilinear parabolic equations with non smooth initial data*, Math. Comput. **49** (1987), 331–357.
- [22] P.E. KLOEDEN, E. PLATEN, NUMERICAL SOLUTION OF STOCHASTIC DIFFERENTIAL EQUATIONS, Applications of Mathematics, **23**, Springer Verlag, New York, 1992.
- [23] M.-N. LE ROUX, *Semidiscretization in Time for Parabolic Problems*, Math. Comput. **33** (1979), 919–931.
- [24] G. LORD, J. ROUGEMONT, *A Numerical Scheme for Stochastic PDEs with Gevrey Regularity*, IMA J. Num. Anal., **24** (2004), n° 4, 587–604.
- [25] A. MILLET, P.L. MORIEN, *On implicit and explicit discretization schemes for parabolic SPDEs in any dimension*, Stochastic Processes and their Applications **115** (2005), no 7, 1073–1106.
- [26] G. N. MILSTEIN, NUMERICAL INTEGRATION OF STOCHASTIC DIFFERENTIAL EQUATIONS, Mathematics and its Applications, **313**, Kluwer Academic Publishers, Dordrecht, 1995.
- [27] G. N. MILSTEIN, M. V. TRETYAKOV, STOCHASTIC NUMERICS FOR MATHEMATICAL PHYSICS, Scientific Computation series, Springer-Verlag, 2004.
- [28] J. PRINTEMS, *On the discretization in time of parabolic stochastic partial differential equations*, Math. Model. and Numer. Anal. **35** (2001), n° 6, 1055–1078.
- [29] T. SHARDLOW, *Numerical methods for stochastic parabolic PDEs*, Numer. Funct. Anal. Optim. **20** (1999), no 1-2, 121–145.
- [30] G. STRANG, G.J. FIX, AN ANALYSIS OF THE FINITE ELEMENT METHOD, Prentice-Hall, Englewood Cliffs, 1973,
- [31] D. TALAY, *Probabilistic numerical methods for partial differential equations: elements of analysis*, Probabilistic models for nonlinear partial differential equations (Montecatini Terme, 1995), 148–196, Lecture Notes in Math., **1627**, Springer, Berlin, 1996.
- [32] V. THOMÉE, GALERKIN FINITE ELEMENT METHODS FOR PARABOLIC PROBLEMS, Springer Verlag, 1997.
- [33] Y. YAN, *Galerkin finite element methods for stochastic parabolic partial differential equations*, SIAM J. Numer. Anal. **43** (2005), n° 4, 1363–1384.
- [34] Y. YAN, *Semidiscrete Galerkin approximation for a linear stochastic parabolic partial differential equation driven by an additive noise*, BIT **44** (2004), n° 4, 829–847.
- [35] J.B. WALSH *Finite element methods for parabolic stochastic PDE's*, Potential Anal. **23** (2005), n° 1, 1–43.

## Appendix.

We now prove the estimate (3.52) on  $M_n$ ,  $n < N - 1$ , defined in (3.51).

We proceed as follows:

$$(3.70) \quad M_n(t) \leq \sup_{1 \leq i \leq I(h)} \left| \frac{F^{N-n-1}(\lambda_{i,h} \Delta t) - e^{-\lambda_{i,h}(T-t_{n+1})}}{1 + \theta \lambda_{i,h} \Delta t} \right| \lambda_{i,h}^{(1-\gamma_1)/2}$$

$$\begin{aligned}
& + \sup_{1 \leq i \leq I(h)} \left| \frac{e^{-\lambda_{i,h}(T-t_{n+1})} - e^{-\lambda_{i,h}(T-t)}}{1 + \theta \lambda_{i,h} \Delta t} \right| \lambda_{i,h}^{(1-\gamma_1)/2} \\
& + \sup_{1 \leq i \leq I(h)} e^{-\lambda_{i,h}(T-t)} \left( 1 - \frac{1}{1 + \theta \Delta t \lambda_{i,h}} \right) \lambda_{i,h}^{(1-\gamma_1)/2} \\
& = a_1 + a_2 + a_3.
\end{aligned}$$

Thanks to (3.49) with  $N$  replaced by  $N - n - 1$  we get

$$\begin{aligned}
a_1 & \leq \frac{\kappa_4}{N - n - 1} \sup_{i \geq 1} \left( \frac{\lambda_{i,h}^{(1-\gamma_1)/2}}{1 + \theta \lambda_{i,h} \Delta t} \right) \\
& \leq \frac{\kappa_4 \Delta t^{(\gamma_1-1)/2}}{(N - n - 1)^{(1-\gamma_1+\gamma)/2}} \sup_{i \geq 1} \left( \frac{(\lambda_{i,h} \Delta t)^{(1-\gamma_1)/2}}{1 + \theta \lambda_{i,h} \Delta t} \right) \\
& \leq \frac{\kappa_4 \Delta t^{\gamma/2}}{((N - n - 1) \Delta t)^{(1-\gamma_1+\gamma)/2}}.
\end{aligned}$$

Indeed, since  $(1 - \gamma_1 + \gamma)/2 < 1$ ,  $(N - n - 1) \geq (N - n - 1)^{(1-\gamma_1+\gamma)/2}$ . In the same way, we have

$$\begin{aligned}
a_2 & \leq \sup_{i \geq 1} \left( \frac{1 - e^{-(t_{n+1}-t)\lambda_{i,h}}}{(1 + \theta \Delta t \lambda_{i,h})} \lambda_{i,h}^{(1-\gamma_1)/2} e^{-(N-n-1)\Delta t \lambda_{i,h}} \right) \\
& \leq C(\gamma) \Delta t^{\gamma/2} \sup_{i \geq 1} \left( \lambda_{i,h}^{(1-\gamma_1+\gamma)/2} e^{-(N-n-1)\Delta t \lambda_{i,h}} \right) \\
& \leq \frac{C(\gamma, \gamma_1) \Delta t^{\gamma/2}}{((N - n - 1) \Delta t)^{(1-\gamma_1+\gamma)/2}},
\end{aligned}$$

where we have used that  $|t - t_{n+1}| \leq \Delta t$  and the inequality  $|e^{-x} - e^{-y}| \leq C_\gamma |x - y|^{\gamma/2}$  and (2.22). Eventually, similar computations lead to

$$\begin{aligned}
a_3 & \leq \Delta t^{\gamma/2} \sup_{i \geq 1} \left( \frac{\theta (\Delta t \lambda_{i,h})^{1-\gamma/2}}{1 + \theta \Delta t \lambda_{i,h}} \lambda_{i,h}^{(1-\gamma_1+\gamma)/2} e^{-(T-t)\lambda_{i,h}} \right) \\
& \leq C(\gamma, \gamma_1) \Delta t^{\gamma/2} (T - t)^{-(1-\gamma_1+\gamma)/2} \\
& \leq C(\gamma, \gamma_1) \Delta t^\gamma ((N - n - 1) \Delta t)^{-(1-\gamma_1+\gamma)/2},
\end{aligned}$$

where we have used again the inequality (2.22).

Gathering these three estimates in (3.70) yields (3.52), for  $n < N - 1$ .